BULLETIN OF THE BRAZILIAN MATHEMATICAL SOCIETY

Bull Braz Math Soc, New Series 36(3), 379-391

On Sobolev infinitesimal rigidity of linear hyperbolic actions on the 2-torus*

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Abstract. Let A be a symmetric hyperbolic matrix in $\mathrm{SL}(2,\mathbb{Z})$ and Γ the subgroup of $\mathrm{SL}(2,\mathbb{Z})$ generated by A. We aim to study the infinitesimal rigidity of the standard action of Γ on the torus $\mathbb{T}^2$. More precisely, we will consider the Sobolev W^s–infinitesimal rigidity of this action (that is to determine if the cohomology space $H^1(\Gamma, W^s(TM))$ is trivial or not), and show that it is W^s–infinitesimally rigid only if $0 \leq s < 1$. A consequence will be that this action is not C^∞–infinitesimally rigid.

Keywords: infinitesimal rigidity, hyperbolic actions.

Mathematical subject classification: 22E40, 57S25, 58F15.

Introduction

Let G be a topological group and Γ be a finitely-generated group. We denote by $R(\Gamma, G)$ the set of all group homomorphisms of Γ into G endowed with the topology of pointwise convergence. If $\gamma_1, \ldots, \gamma_k$ are fixed generators of Γ, one may consider $R(\Gamma, G)$ as a closed subset of G^k by means of the map $\rho \mapsto (\rho(\gamma_1), \ldots, \rho(\gamma_k))$. Note that G acts naturally on $R(\Gamma, G)$ by conjugation: if $\rho \in R(\Gamma, G)$ and $g \in G$, then for all $\gamma \in \Gamma$, $(g.\rho)(\gamma) = g\rho(\gamma)g^{-1}$. A homomorphism ρ_0 is said to be *locally rigid* if its orbit is open in $R(\Gamma, G)$, or equivalently, if there exists a neighborhood U of ρ_0 in $R(\Gamma, G)$ such that every $\rho \in U$ is conjugated to ρ_0. In the case G is a Lie group, G acts differentiably on itself by conjugation (for any g_0 in G, $\Phi_{g_0} : g \mapsto g_0 g {g_0}^{-1}$ is an automorphism of G), and this action induces an action of G on its Lie Algebra $\mathfrak{g}$, which is isomorphic to the tangent space T_1G, by means of the derivatives $\Phi'_{g_0}(1)$. This is what is called the *adjoint representation* Ad_G of G in $\mathfrak{g}$. In this context,

Received 17 June 2004.

*I would like to thank A. El Kacimi for introducing me this problem about which we had many fruitful discussions.

Weil [W] proved that if $\rho \in R(\Gamma, G)$ is such that $H^1(\Gamma, \mathrm{Ad}_G \circ \rho) = 0$ then ρ is locally rigid (see also Raghunathan [R], chapter VI).

Let now M be a compact C^∞ manifold and $\mathrm{Diff}(M)$ be the group of C^∞ diffeomorphisms of M endowed with the usual C^∞ topology. We denote by $C^\infty(TM)$ the space of C^∞ vector fields on M. Any representation $\rho \in R(\Gamma, \mathrm{Diff}(M))$ induces a linear action of Γ on $C^\infty(TM)$, and ρ is said to be *C^∞–infinitesimally rigid* (or for short *infinitesimally rigid*) if $H^1(\Gamma, C^\infty(TM)) = 0$. This terminology used by Zimmer [Z] suggests an analogy with Weil's theorem. Indeed, $C^\infty(TM)$ is the Lie algebra of the infinite dimensional Lie group $\mathrm{Diff}(M)$, and the natural action of Γ on $C^\infty(TM)$ is in fact the composition of the representation of Γ into $\mathrm{Diff}(M)$ with the adjoint representation of $\mathrm{Diff}(M)$ on its Lie algebra. Nevertheless, there is no established results connecting infinitesimal and local rigidity.

For results about local rigidity of the standard action of $\mathrm{SL}(n, \mathbb{Z})$ on the torus $\mathbb{T}^n$, the reader can refer to [H1], [H2], [KL], [KLZ]. With regard to infinitesimal rigidity of these actions on tori, many results have been established, especially when $n \geq 3$ and Γ is a subgroup of finite index in $\mathrm{SL}(n, \mathbb{Z})$. For instance, Pollicott [P] showed that the action of $\mathrm{SL}(3, \mathbb{Z})$ on $\mathbb{T}^3$ is infinitesimally rigid, and Lewis [Le] proved that for $n \geq 7$ and Γ a subgroup of finite index in $\mathrm{SL}(n, \mathbb{Z})$, the action of Γ on $\mathbb{T}^n$ is also infinitesimally rigid. A more general result is given by Hurder [H3], stating that for $n \geq 3$ and Γ a subgroup of finite index in $\mathrm{SL}(n, \mathbb{Z})$, every affine action of Γ on $\mathbb{T}^n$ associated to the standard action is infinitesimally rigid. The reader interested by affine actions on tori can also refer to [Lu]. As a matter of fact the case $n = 2$ is raised in [H1], [H2], but these results only concern local rigidity.

The goal of this note is to study infinitesimal rigidity for the following example: let A be a symmetric and hyperbolic matrix (that is to say that A has no eigenvalue of modulus 1, or equivalently in this case, that $|\operatorname{tr} A| > 2$) in $\mathrm{SL}(2, \mathbb{Z})$ acting linearly on $M = \mathbb{T}^2$. This matrix has two irrational eigenvalues, say λ and λ^{-1} with $|\lambda| > 1$. The infinite cyclic subgroup Γ generated by A is of infinite index in $\mathrm{SL}(2, \mathbb{Z})$ and its action on M is Anosov. Let $L^2(TM)$ be the Hilbert space of square integrable vector fields on M and, for every real number $s \geq 0$, denote by $W^s(TM)$ the space of vector fields of s–Sobolev class; naturally $W^0(TM) = L^2(TM)$ and the intersection $W^\infty(TM)$ of all the $W^s(TM)$ is equal to $C^\infty(TM)$. We will show that the action of Γ on M is W^s–infinitesimally rigid if and only if $s < 1$, and in particular, that the action of Γ on M is not C^∞–infinitesimally rigid.

1 The space $H^0(\Gamma, L^2(TM))$ of invariant L^2 vector fields

The problem of infinitesimal rigidity of the action of Γ on M, that is to determine $H^1(\Gamma, C^\infty(TM))$, makes sense when considering real vector fields on M. However, the special case of $M = \mathbb{T}^2$ will enable us to use Fourier analysis and then to treat the more general problem of the cohomology of Γ acting on complex vector fields on M.

The local coordinates of a particular point $z \in M = \mathbb{T}^2$ will be denoted by (x, y). If X is a vector field on M, then on the covering space $\mathbb{R}^2$, we have

$$X(z) = f(x, y)\frac{\partial}{\partial x} + g(x, y)\frac{\partial}{\partial y} ,$$

where f and g are $\mathbb{Z}^2$-periodic functions. The field X is said to be L^2 if the coefficients f and g are L^2 functions on M. In this case, f and g have the following Fourier expansions:

$$f(x, y) = \sum_{m,n} f_{m,n} e^{2\imath\pi(mx+ny)} \ , \ g(x, y) = \sum_{m,n} g_{m,n} e^{2\imath\pi(mx+ny)} ,$$

where $(f_{m,n})$ and $(g_{m,n})$ are elements of the Hilbert space $\ell^2(\mathbb{Z}^2, \mathbb{C})$ of complex square summable families indexed by $\mathbb{Z}^2$. In the case X is a real vector field, coefficients $f_{m,n}$ and $g_{m,n}$ verify additional relations:

$$f_{-m,-n} = \overline{f_{m,n}} \quad \text{and} \quad g_{-m,-n} = \overline{g_{m,n}}.$$

From now on, we identify L^2 functions on M with elements of $\ell^2(\mathbb{Z}^2, \mathbb{C})$.

Since $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ is symmetric, it admits a diagonalization in an orthogonal basis of $\mathbb{R}^2$, say

$$A = P \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} P^{-1} \quad \text{with} \quad P = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in \mathrm{SO}(2, \mathbb{R}).$$

The action of $\Gamma =< A >$ on $L^2(TM)$ is given by

$$A_* X(Az) = (af(z) + bg(z))\frac{\partial}{\partial x} + (bf(z) + cg(z))\frac{\partial}{\partial y}.$$

Linear eigenvector fields $X_\lambda = \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y}$ and $X_{\lambda^{-1}} = -\sin\theta \frac{\partial}{\partial x} + \cos\theta \frac{\partial}{\partial y}$ respectively associated to λ and λ^{-1} (that is to say $A_* X_\lambda = \lambda X_\lambda$ and

$A_*X_{\lambda^{-1}} = \lambda^{-1}X_{\lambda^{-1}}$) form a basis of $L^2(TM)$ in which any vector field X is written $X(z) = u(z)X_\lambda + v(z)X_{\lambda^{-1}}$, where u and v are L^2 functions on M. Then we have

$$A_*X\,(Az) = \lambda u(z)X_\lambda + \lambda^{-1}v(z)X_{\lambda^{-1}}. \tag{1}$$

For convenience, we put for any $(m, n) \in \mathbb{Z}^2$:

$$\begin{pmatrix}\overline{m}\\ \overline{n}\end{pmatrix} = A\begin{pmatrix}m\\ n\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}\underline{m}\\ \underline{n}\end{pmatrix} = A^{-1}\begin{pmatrix}m\\ n\end{pmatrix}.$$

If (m_0, n_0) is a reference pair in $\mathbb{Z}^2$, we put for any $k \in \mathbb{Z}$:

$$\begin{pmatrix}m_k\\ n_k\end{pmatrix} = A^k\begin{pmatrix}m_0\\ n_0\end{pmatrix}.$$

We can now show that

Theorem 1.1. *The space $H^0(\Gamma, L^2(TM))$ of Γ–invariant L^2 vector fields is trivial.*

Proof. An L^2 vector field X is invariant if and only if $A_*X = X$; this is equivalent to

$$\forall\, z \in M,\ u(Az) = \lambda u(z) \quad \text{and} \quad v(Az) = \lambda^{-1}v(z).$$

Replacing for instance the function u by its Fourier expansion, we obtain:

$$\sum_{p,q} u_{p,q}e^{2\imath\pi((ap+bq)x+(bp+cq)y)} = \sum_{m,n} \lambda\, u_{m,n}e^{2\imath\pi(mx+ny)}$$

i.e.
$$\sum_{p,q} u_{p,q}e^{2\imath\pi(\overline{p}x+\overline{q}y)} = \sum_{m,n} \lambda\, u_{m,n}e^{2\imath\pi(mx+ny)}.$$

By identifying coefficients, and doing it similarly for v, we obtain

$$u_{\underline{m},\underline{n}} = \lambda u_{m,n} \text{ and } v_{\underline{m},\underline{n}} = \lambda^{-1}v_{m,n}.$$

Using these conditions, we can first deduce that $u_{0,0} = v_{0,0} = 0$.

Suppose now that there exists a pair $(m_0, n_0) \neq 0$ such that $u_{m_0,n_0} \neq 0$. Then for all $k \in \mathbb{N}$, $u_{m_{-k},n_{-k}} = \lambda^k u_{m_0,n_0}$; hence $\lim_{k\to-\infty}|u_{m_k,n_k}| = +\infty$. But this contradicts the fact that the orbit by A of every point of $\mathbb{Z}^2$ different from 0 is infinite (see lemma 1.3 below), and that the Fourier coefficient $u_{m,n}$ of u tends to 0 when (m, n) tends to infinity. Thus we have $u_{m,n} = 0$ (and in the same manner $v_{m,n} = 0$) for all $(m, n) \in \mathbb{Z}^2$. So $u = v = 0$ *i.e.* $X = 0$. $\square$

We give from now on two simple lemmas concerning the orbits of $\mathbb{Z}^2$ for the action of A that will be useful especially in the last section of this paper. The first one is about the number of these orbits, and the second one specifies their asymptotic behaviour.

Lemma 1.2. *There is a countable infinite number of orbits for the action of A on $\mathbb{Z}^2$.*

Proof. It suffices to remark that two pairs of integers that belong to the same orbit have the same G.C.D, so each point of the form (p, p) where p is a prime number belongs to a unique orbit. $\square$

Lemma 1.3. *For any fixed pair $(m_0, n_0) \neq 0$, there exists positive constants c_+ and c_- such that*

$$m_k^2 + n_k^2 \underset{+\infty}{\sim} c_+\lambda^{2k} \quad \text{and} \quad m_k^2 + n_k^2 \underset{-\infty}{\sim} c_-\lambda^{-2k}$$

Proof. Since $A = P \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} P^{-1}$ with $P = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, we have:

$$\begin{aligned} \begin{pmatrix} m_k \\ n_k \end{pmatrix} &= P \begin{pmatrix} \lambda^k & 0 \\ 0 & \lambda^{-k} \end{pmatrix} P^{-1} \begin{pmatrix} m_0 \\ n_0 \end{pmatrix} \\ &= \begin{pmatrix} \lambda^k \cos\theta(m_0\cos\theta + n_0\sin\theta) + \lambda^{-k}\sin\theta(-m_0\sin\theta + n_0\cos\theta) \\ -\lambda^k \sin\theta(m_0\cos\theta + n_0\sin\theta) + \lambda^{-k}\cos\theta(-m_0\sin\theta + n_0\cos\theta) \end{pmatrix}. \end{aligned}$$

So

$$m_k{}^2 + n_k{}^2 = \lambda^{2k}(m_0\cos\theta + n_0\sin\theta)^2 + \lambda^{-2k}(-m_0\sin\theta + n_0\cos\theta)^2$$

thus

$$m_k^2 + n_k^2 \underset{+\infty}{\sim} c_+\lambda^{2k}\ ,\ m_k^2 + n_k^2 \underset{-\infty}{\sim} c_-\lambda^{-2k},$$

where

$$c_+ = (m_0\cos\theta + n_0\sin\theta)^2 \quad \text{and} \quad c_- = (-m_0\sin\theta + n_0\cos\theta)^2.$$

Now $(\cos\theta, \sin\theta)$ is an eigenvector of A and we can show then that $\cos\theta$ and $\sin\theta$ are rationally independent. So $c_+ > 0$ and $c_- > 0$. $\square$

2 W^s-infinitesimal rigidity for $0 \leq s < 1$

For every real $s \geq 0$, we say that a vector field X is in $W^s(TM)$, the s-Sobolev space of vector fields on M, if its coefficients are both in $W^s(M)$, with

$$W^s(M) = \left\{ f : \mathbb{Z}^2 \to \mathbb{C} \;\middle|\; \sum_{(m,n)\neq 0} (m^2+n^2)^s |f_{m,n}|^2 < +\infty \right\}.$$

Then for $0 \leq s \leq s'$, we have

$$C^\infty(M) = \bigcap_{s\geq 0} W^s(M) \subset W^{s'}(M) \subset W^s(M) \subset W^0(M) = L^2(M).$$

For every $s < \infty$, $W^s(M)$ is a Hilbert space with the hermitian product $\langle \bullet, \bullet \rangle_s$:

$$\langle f, g \rangle_s = f_{0,0}\overline{g_{0,0}} + \sum_{(m,n)\neq 0} (m^2+n^2)^s f_{m,n}\overline{g_{m,n}},$$

and the associated norm :

$$||f||_s = \left(|f_{0,0}|^2 + \sum_{(m,n)\neq 0} (m^2+n^2)^s |f_{m,n}|^2 \right)^{\frac{1}{2}}.$$

Let $\{\delta^{m,n}\}_{(m,n)\in\mathbb{Z}^2}$ be the canonical Hilbert basis of $\ell^2(\mathbb{Z}^2, \mathbb{C})$, that is:

$$\delta^{m,n}_{p,q} = \begin{cases} 1 & \text{if } (p,q) = (m,n), \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that $\{\delta^{m,n}\}$ is a Hilbert basis of $W^s(M)$ for any $s \geq 0$.

Let $X \in W^s(TM)$ with $X(Z) = u(z)X_\lambda + v(z)Y_{\lambda^{-1}}$. According to expression (1), we have $A_*X(z) = U(z)X_\lambda + V(z)X_{\lambda^{-1}}$, with

$$\begin{aligned} U(z) &= \lambda u(A^{-1}z) = \lambda \sum_{m,n} u_{m,n} e^{2\imath\pi(m(cx-by)+n(-bx+ay))} \\ &= \lambda \sum_{m,n} u_{m,n} e^{2\imath\pi(\underline{m}x+\underline{n}y)} = \lambda \sum_{m,n} u_{\overline{m},\overline{n}} e^{2\imath\pi(mx+ny)}; \\ V(z) &= \lambda^{-1} v(A^{-1}z) = \lambda^{-1} \sum_{m,n} v_{\overline{m},\overline{n}} e^{2\imath\pi(mx+ny)}. \end{aligned}$$

So $U = \lambda T_A u$ and $V = \lambda^{-1} T_A v$, where T_A is the linear operator consisting of the permutation of the indices of Fourier coefficients by A:

$$\forall\, f \in L^2(M),\; (T_A f)_{m,n} = f_{\overline{m},\overline{n}}.$$

It is clear that T_A is a bijective isometry of $\ell^2(\mathbb{Z}^2, \mathbb{C})$ and that ${T_A}^{-1} = T_{A^{-1}}$.

Let us be more precise with the the following proposition:

Proposition 2.1. *For any $s \geq 0$, T_A and $T_{A^{-1}}$ are bijective continuous operators of $W^s(M)$ and $|||T_{A^{-1}}|||_s = |||T_A|||_s = |\lambda|^s$.*

Proof. Let $f = \sum_{m,n} f_{m,n}\delta^{m,n} \in W^s(M)$; then $T_A f = \sum_{m,n} f_{\overline{m},\overline{n}}\delta^{m,n}$ and

$$\begin{aligned} ||T_A f||_s^2 &= |f_{0,0}|^2 + \sum_{(m,n)\neq 0} (m^2+n^2)^s |f_{\overline{m},\overline{n}}|^2 \\ &= |f_{0,0}|^2 + \sum_{(m,n)\neq 0} (\underline{m}^2+\underline{n}^2)^s |f_{m,n}|^2. \end{aligned}$$

Since A^{-1} is diagonalizable in an orthogonal basis, we can easily show that, as a linear operator of the usual euclidian space $\mathbb{R}^2$, $|||A^{-1}||| = |\lambda| > 1$. Consequently,

$$\begin{aligned} ||T_A f||_s^2 &\leq |f_{0,0}|^2 + \sum_{(m,n)\neq 0} |||A^{-1}|||^{2s} (m^2+n^2)^s |f_{m,n}|^2 \\ &\leq \lambda^{2s} ||f||_s^2, \end{aligned}$$

and this for any $f \in W^s(M)$, hence $|||T_A|||_s \leq |\lambda|^s$.

Let $\epsilon > 0$; the function $x \mapsto x^s$ is continuous in $]0, +\infty[$, therefore there exists $\eta > 0$ such that

$$\forall\, x > 0,\ |x - |\lambda|| \leq \eta \Rightarrow |x^s - |\lambda|^s| \leq \epsilon.$$

Then, for a small enough $\eta > 0$:

$$|\lambda|^s - \epsilon \leq (|\lambda| - \eta)^s.$$

Now, as $|||A^{-1}||| = |\lambda|$, there exists a pair $(p, q) \in \mathbb{Z} \times \mathbb{N}^*$ such that

$$|\lambda| - \eta \leq ||A^{-1}(p(p^2+q^2)^{-\frac{1}{2}}, q(p^2+q^2)^{-\frac{1}{2}})|| \leq |\lambda|,$$

and so

$$(p^2+q^2)^s(|\lambda| - \eta)^{2s} \leq (\underline{p}^2 + \underline{q}^2)^s \leq (p^2+q^2)^s \lambda^{2s}.$$

Let $g = (p^2+q^2)^{-\frac{s}{2}}\delta^{p,q} \in W^s(M)$, so that $||g||_s = 1$ and

$$(|\lambda| - \eta)^{2s} \leq ||T_A g||_s^2 = (\underline{p}^2 + \underline{q}^2)^s (p^2+q^2)^{-s} \leq \lambda^{2s},$$

and then

$$|\lambda|^s - \epsilon \leq (|\lambda| - \eta)^s \leq ||T_A g||_s \leq |\lambda|^s.$$

Thus,

$$\forall\, \epsilon > 0,\ \exists\, g \in W^s(M),\ ||g||_s = 1 \quad \text{and} \quad |\lambda|^s - \epsilon \leq ||T_A g||_s \leq |\lambda|^s,$$

and we can conclude that $|||T_A|||_s = |\lambda|^s$. Moreover, all this holds for A^{-1} instead of A, so that we also have $|||T_{A^{-1}}|||_s = |\lambda|^s$. □

We can then assert:

Corollary 2.2. *For any $s \geq 0$, $W^s(TM)$ is a sub–Γ–module of $L^2(TM)$.*

Since the group Γ is isomorphic to $\mathbb{Z}$, it is well known that

$$H^1(\Gamma, W^s(TM)) = W^s(TM) \Big/ \left\{ A_* X - X \mid X \in W^s(TM) \right\}.$$

To prove that $H^1(\Gamma, W^s(TM)) = 0$ is equivalent to show that, for each vector field Y in $W^s(TM)$, there exists a vector field X in $W^s(TM)$ such that $A_*X - X = Y$. Let $X(z) = u(z)X_\lambda + v(z)X_{\lambda^{-1}}$ and $Y(z) = U(z)X_\lambda + V(z)X_{\lambda^{-1}}$; then the equation $A_*X - X = Y$ is equivalent to the system

$$\left\{ \begin{array}{rcl} (\lambda T_A - \mathrm{Id})u & = & U \\ (\lambda^{-1} T_A - \mathrm{Id})v & = & V \end{array} \right. \tag{2}$$

or equivalently:

$$\left\{ \begin{array}{rcl} (T_{A^{-1}} - \lambda \mathrm{Id})u & = & -T_{A^{-1}} U \\ (T_A - \lambda \mathrm{Id})v & = & \lambda V \end{array} \right. \tag{3}$$

Let us put $S_A = T_A - \lambda\mathrm{Id}$ and $S_{A^{-1}} = T_{A^{-1}} - \lambda\mathrm{Id}$.

We can now easily prove that

Proposition 2.3. *For* $0 \leq s < 1$, $H^1(\Gamma, W^s(TM)) = 0$.

Proof. It is an established fact that, for a continuous operator of a Hilbert space (and more generally of a Banach space), its spectral radius is not greater than its norm. So, if $0 \leq s < 1$, then $|\lambda|^s < |\lambda|$ and as a consequence, λ is a regular value of both T_A and $T_{A^{-1}}$, that is to say S_A and $S_{A^{-1}}$ are invertible operators of $W^s(M)$. Hence there exists a unique pair of functions u and v verifying system (3), and so the equation $A_*X - X = Y$ has a unique solution in $W^s(M)$. □

It is straightforward that T_A and $T_{A^{-1}}$, and then S_A and $S_{A^{-1}}$, sends real-valued functions to real-valued functions. It is not much more difficult to show it for their inverses S_A^{-1} and $S_{A^{-1}}^{-1}$. So we can assert:

Theorem 2.4. *For* $0 \leq s < 1$, *the action of* Γ *on* M *is* W^s*–infinitesimally rigid.*

3 The space $H^1(\Gamma, W^s(TM))$ for $s \geq 1$

Now, let us show that, for $s \geq 1$, the vector space $H^1(\Gamma, W^s(TM))$ is non trivial. In order to do that, we just have to exhibit a vector field Y in $W^s(TM)$ such that the unique solution X in $L^2(TM)$ of the equation $A_*X - X = Y$ is not in $W^s(TM)$. It is then useful to determine the preimages of the basis $\{\delta^{m,n}\}$ by $S_{A^{-1}}$.

Proposition 3.1. *For any fixed pair* $(m_0, n_0) \in \mathbb{Z}^2$, *we set* $\eta^{m_0,n_0} = S_{A^{-1}}^{-1}\delta^{m_0,n_0}$. *Then*

$$\eta^{0,0} = \frac{1}{1-\lambda}\delta^{0,0} \quad \text{and} \quad \eta^{m_0,n_0} = -\sum_{k=0}^{+\infty} \lambda^{-k-1}\delta^{m_k,n_k} \quad \text{if} \quad (m_0, n_0) \neq 0.$$

Proof. By using the relation $\eta^{m_0,n_0}_{\underline{m},\underline{n}} - \lambda\eta^{m_0,n_0}_{m,n} = \delta^{m_0,n_0}_{m,n}$, we have:

- If $(m_0, n_0) = 0$, then

$$\begin{cases} \eta^{0,0}_{0,0} - \lambda\eta^{0,0}_{0,0} = 1 \\ \eta^{0,0}_{\underline{m},\underline{n}} - \lambda\eta^{0,0}_{m,n} = 0 \quad \text{if } (m,n) \neq 0; \end{cases}$$

  hence

$$\begin{cases} \eta^{0,0}_{0,0} = \dfrac{1}{1-\lambda}, \\ \eta^{0,0}_{\underline{m},\underline{n}} = \lambda\eta^{0,0}_{m,n} \qquad \text{if } (m,n) \neq 0. \end{cases}$$

  If we refer to the proof of theorem 1.1, we know that for $(m, n) \neq 0$, we necessarily have $\eta^{0,0}_{m,n} = 0$, hence

$$\eta^{0,0} = \frac{1}{1-\lambda}\delta^{0,0}.$$

- If $(m_0, n_0) \neq 0$, then

$$\begin{cases} \eta^{m_0,n_0}_{m_{-1},n_{-1}} - \lambda\eta^{m_0,n_0}_{m_0,n_0} = 1 \\ \eta^{m_0,n_0}_{\underline{m},\underline{n}} - \lambda\eta^{m_0,n_0}_{m,n} = 0 \qquad \text{if } (m,n) \neq (m_0,n_0). \end{cases}$$

  Thus

$$\begin{aligned} \eta^{m_0,n_0}_{m_{-1},n_{-1}} &= 1 + \lambda\eta^{m_0,n_0}_{m_0,n_0}, \\ \eta^{m_0,n_0}_{m_{-2},n_{-2}} &= \lambda\eta^{m_0,n_0}_{m_{-1},n_{-1}} = \lambda(1 + \lambda\eta^{m_0,n_0}_{m_0,n_0}), \\ &\vdots \\ \eta^{m_0,n_0}_{m_{-k},n_{-k}} &= \lambda^{k-1}(1 + \lambda\eta^{m_0,n_0}_{m_0,n_0}) \quad \text{for all} \quad k \geq 1. \end{aligned}$$

Similarly, in the opposite direction, we have also

$$\begin{aligned} \eta^{m_0,n_0}_{m_1,n_1} &= \lambda^{-1}\eta^{m_0,n_0}_{m_0,n_0}\,, \\ \eta^{m_0,n_0}_{m_2,n_2} &= \lambda^{-2}\eta^{m_0,n_0}_{m_0,n_0}\,, \\ &\vdots \\ \eta^{m_0,n_0}_{m_k,n_k} &= \lambda^{-k}\eta^{m_0,n_0}_{m_0,n_0} \quad \text{for all} \quad k \geq 0\,, \end{aligned}$$

Once again because $\lim_{k\to+\infty} \eta^{m_0,n_0}_{m_{-k},n_{-k}} = 0$, we necessarily have $1 + \lambda\eta^{m_0,n_0}_{m_0,n_0} = 0$, and consequently

$$\eta^{m_0,n_0}_{m_k,n_k} = \begin{cases} 0 & \text{if } k < 0, \\ -\lambda^{-k-1} & \text{if } k \geq 0. \end{cases}$$

When the point (m, n) does not belong to the orbit of (m_0, n_0), we use the same argument again to assert that $\eta^{m_0,n_0}_{m,n} = 0$. Finally, for all $(m_0, n_0) \neq 0$:

$$\eta^{m_0,n_0} = -\sum_{k=0}^{+\infty} \lambda^{-k-1}\delta^{m_k,n_k}.$$

We verify by the way that η^{m_0,n_0} is actually in $W^0(M) = L^2(M)$ and that

$$||\eta^{m_0,n_0}||_0 = \frac{|\lambda|}{\sqrt{1-\lambda^{-2}}}. \qquad \square$$

For every pair (m_0, n_0), it is clear that δ^{m_0,n_0} is in $W^s(M)$ for any real s; but what about η^{m_0,n_0} (of course in the case $(m_0, n_0) \neq 0$, since it is obvious that $\eta^{0,0}$ is in every $W^s(M)$)?

Proposition 3.2. *For any pair $(m_0, n_0) \neq 0$, η^{m_0,n_0} is not in $W^s(M)$ if $s \geq 1$.*

Proof. We have to show that the series $\sum_{k\geq 0}(m_k^2 + n_k^2)^s\lambda^{-2k-2}$ is divergent. Lemma 1.3 implies that $(m_k^2 + n_k^2)^s\lambda^{-2k-2} \underset{+\infty}{\sim} \lambda^{-2}{c_+}^s\lambda^{2(s-1)k}$, and the series $\sum_{k\geq 0}(m_k^2 + n_k^2)^s\lambda^{-2k-2}$ is convergent if and only if $s < 1$. $\square$

Finally, we are in a position to give an example of a countable family $\{Y_p\}$ of real vector fields on M such that $Y_p \in W^s(TM)$ for any $s \geq 0$ and for which the unique field X_p in $L^2(TM)$ such that $A_*X_p - X_p = Y_p$ is not in $W^s(TM)$ if $s \geq 1$.

Proposition 3.3. *Let $\mathcal{P}$ be the set of prime numbers and put for every $p \in \mathcal{P}$,*

$$Y_p(z) = 2\cos((a+b)px + (b+c)py)X_\lambda.$$

Then for any $p \in \mathcal{P}$, the unique vector field $X_p \in L^2(TM)$ such that

$$A_* X_p - X_p = Y_p$$

is not in $W^s(TM)$ for $s \geq 1$.

Proof. It is obvious that $Y_p = U_p X_\lambda$ with $U_p = \delta^{-\overline{p},-\overline{p}} + \delta^{\overline{p},\overline{p}}$ is in $W^s(TM)$ for any $s \geq 0$. Now $X_p = u_p X_\lambda$ with $u_p = -S_{A^{-1}}^{-1} T_{A^{-1}} U_p = -(\eta^{-p,-p} + \eta^{p,p})$, and if we set $(m_0, n_0) = (p, p)$, we have

$$u_p = \sum_{k=0}^{+\infty} \lambda^{-k-1} \delta^{-m_k,-n_k} + \sum_{k=0}^{+\infty} \lambda^{-k-1} \delta^{m_k,n_k},$$

hence

$$||u_p||_s^2 = 2\sum_{k=0}^{+\infty} (m_k^2 + n_k^2)^s \lambda^{-2k-2} = 2||\eta^{p,p}||_s^2 = +\infty.$$

So $u_p \notin W^s(M)$ and then $X_p \notin W^s(TM)$. □

Proposition 3.4. *The family $\{[Y_p]\}_{p\in\mathcal{P}}$ is linearly independent in $H^1(\Gamma, W^s(TM))$ when $s \geq 1$.*

Proof. Let J be a finite subset of $\mathcal{P}$ and $(\mu_p)_{p\in J} \in \mathbb{C}^J$ such that

$$\sum_{p\in J} \mu_p [Y_p] = 0 \quad \text{in} \quad W^s(TM),\ s \geq 1.$$

Then, there exists $X \in W^s(TM)$, $X = uX_\lambda$, such that $\sum_{p\in J} \mu_p Y_p = A_* X - X$ with $u = \sum_{p\in J} \mu_p u_p$, and

$$||u||_s^2 = \sum_{p\in J} |\mu_p|^2 ||u_p||_s^2 \geq |\mu_q|^2 ||u_q||_s^2 \quad \text{for any } q \in J.$$

So, since $||u||_s^2 < +\infty$ and $||u_q||_s^2 = +\infty$, we necessarily have $\mu_q = 0$ for any $q \in J$. □

As a conclusion, we have:

Theorem 3.5. *The action of* Γ *on* $M = \mathbb{T}^2$ *is* W^s*–infinitesimally rigid if and only if* $0 \leq s < 1$. *Moreover, in case* $1 \leq s \leq \infty$, *the space* $H^1(\Gamma, W^s(TM))$ *is infinite-dimensionnal.*

We could expect to obtain similar results for $A \in \mathrm{SL}(n, \mathbb{Z})$ acting on $\mathbb{T}^n$ with $n > 2$ using the same kind of method. However, in dimension 2, the fact that the two eigenvalues of A are the inverse one of the other appears to be essential. So a generalization in higher dimension seems to be difficult, except maybe in the case n is even, because there exist then symmetric hyperbolic matrices A in $\mathrm{SL}(n, \mathbb{Z})$ such that A and A^{-1} have the same eigenvalues.

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